Programme 11

Partial differential equations

Frames 1 to 66

Learning outcomes

When you have completed this Programme you will be able to:

- Summarise the introductory methods of solving ordinary differential equations
- Solve partial differential equations that are amenable to solution by direct integration
- Apply initial and boundary conditions
- Solve the one-dimensional wave and heat equations by separating the variables and obtaining eigenfunctions and corresponding eigenvalues
- Solve the two-dimensional Laplace equation in Cartesian coordinates
- Recognise the need for alternative coordinate systems and solve the two-dimensional Laplace equation in plane polar coordinates

Prerequisite: Engineering Mathematics (Fifth Edition)
Programmes 24 First-order differential equations and 25 Second-order differential equations

Introduction

1

The formation of ordinary linear differential equations and their solution by various methods were covered in some detail in Programmes 24 and 25 of *Engineering Mathematics (Fifth Edition)*, and reference to these before undertaking the new work of this Programme could be beneficial – especially Programme 25 which dealt with second-order equations. Working through the Test exercise of that Programme would provide worthwhile revision.

The main results obtained are listed here for convenience and easy reference.

1 Equations of the form $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$

Auxiliary equation $am^2 + bm + c = 0$. Solutions depend on the roots of this equation.

(a) Real and different roots: $m = m_1$ and $m = m_2$

$$\text{Solution } y = Ae^{m_1x} + Be^{m_2x} \qquad (1)$$

(b) Real and equal roots: $m = m_1$ (twice)

$$\text{Solution } y = e^{m_1x}(A + Bx) \qquad (2)$$

(c) Complex roots: $m = \alpha \pm j\beta$

$$\text{Solution } y = e^{\alpha x}(A\cos\beta x + B\sin\beta x) \qquad (3)$$

2 Equations of the form $\dfrac{d^2y}{dx^2} \pm n^2y = 0$

(a) $\dfrac{d^2y}{dx^2} + n^2y = 0 \quad \therefore \; m^2 + n^2 = 0 \quad \therefore \; m^2 = -n^2 \quad \therefore \; m = \pm jn$

$$\text{Solution } y = A\cos nx + B\sin nx \qquad (4)$$

(b) $\dfrac{d^2y}{dx^2} - n^2y = 0 \quad \therefore \; m^2 - n^2 = 0 \quad \therefore \; m^2 = n^2 \quad \therefore \; m = \pm n$

$$\left.\begin{aligned} \text{Solution } y &= A\cosh nx + B\sinh nx \\ \text{or } y &= Ae^{nx} + Be^{-nx} \\ \text{or } y &= A\sinh n(x + \phi) \end{aligned}\right\} \qquad (5)$$

In each case, A and B are arbitrary constants depending on the initial conditions, and in the last form ϕ is an arbitrary constant.

Partial differential equations

2

A partial differential equation is a relationship between a dependent variable u and two or more independent variables $(x, y, t, \ldots)$ and partial derivatives of u with respect to these independent variables. The solution is therefore of the form $u = f(x, y, t, \ldots)$.

Solution by direct integration

The simplest form of partial differential equation is such that a solution can be determined by direct partial integration.

Example 1

Solve the equation $\dfrac{\partial^2 u}{\partial x^2} = 12x^2(t+1)$ given that at $x = 0$, $u = \cos\ 2t$ and $\dfrac{\partial u}{\partial x} = \sin t$. Notice that the boundary conditions are functions of t and not just constants. $\dfrac{\partial^2 u}{\partial x^2} = 12x^2(t+1)$. Integrating partially with respect to x, we have $\dfrac{\partial u}{\partial x} = 4x^3(t+1) + \phi(t)$ where the arbitrary function $\phi(t)$ takes the place of the normal arbitrary constant in ordinary integration. Integrating partially again with respect to x gives

$u = \ldots\ldots\ldots\ldots$

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$$\boxed{u = x^4(t+1) + x\phi(t) + \theta(t)}$$

where $\theta(t)$ is a second arbitrary function.

To find the two arbitrary functions $\phi(t)$ and $\theta(t)$, we apply the given initial conditions that at $x = 0$, $\dfrac{\partial u}{\partial x} = \sin t$ and $u = \cos\ 2t$. Substituting these in the relevant equations gives

$\phi(t) = \ldots\ldots\ldots\ldots;\ \theta(t) = \ldots\ldots\ldots\ldots$

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$$\boxed{\phi(t) = \sin t;\ \ \theta(t) = \cos 2t}$$

Because

$$u = x^4(t+1) + x\sin t + \cos 2t$$

Example 2

Solve the equation $\dfrac{\partial^2 u}{\partial x \partial y} = \sin(x+y)$, given that at $y = 0$, $\dfrac{\partial u}{\partial x} = 1$ and at $x = 0$, $u = (y-1)^2$.

In just the same way as before, $u = \ldots\ldots\ldots\ldots$

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$$\boxed{u = -\sin(x+y) + x + \sin x + (y-1)^2}$$

Because

$$\frac{\partial^2 u}{\partial x \partial y} = \sin(x+y) \quad \therefore \quad \frac{\partial u}{\partial x} = -\cos(x+y) + \phi(x).$$

At $y = 0$, $\dfrac{\partial u}{\partial x} = 1 \quad \therefore \quad 1 = -\cos x + \phi(x) \quad \therefore \quad \phi(x) = 1 + \cos x$

$$\therefore \quad \frac{\partial u}{\partial x} = -\cos(x+y) + 1 + \cos x$$

Integrating again partially, this time with respect to x, we have

$$u = -\sin(x+y) + x + \sin x + \theta(y)$$

But at $x = 0$, $u = (y-1)^2$. $\therefore \quad (y-1)^2 = -\sin y + \theta(y)$

$$\therefore \quad \theta(y) = (y-1)^2 + \sin y$$

$$\therefore \quad u = -\sin(x+y) + x + \sin x + \sin y + (y-1)^2$$

Initial conditions and boundary conditions

As with any differential equation, the arbitrary constants or arbitrary functions in any particular case are determined from the additional information given concerning the variables of the equation. These extra facts are called the *initial conditions* or, more generally, the *boundary conditions* since they do not always refer to zero values of the independent variables.

Example 3

Solve the equation $\dfrac{\partial^2 u}{\partial x \partial y} = \sin x \cos y$, subject to the boundary conditions that at $y = \dfrac{\pi}{2}$, $\dfrac{\partial u}{\partial x} = 2x$ and at $x = \pi$, $u = 2\sin y$.

Work through it: it is easy enough. $u = \ldots\ldots\ldots\ldots$

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$$\boxed{u = x^2 + \cos x(1 - \sin y) + \sin y + 1 - \pi^2}$$

Because

$$\frac{\partial^2 u}{\partial x \partial y} = \sin x \cos y \qquad \therefore \quad \frac{\partial u}{\partial x} = \sin x \sin y + \phi(x)$$

But $\dfrac{\partial u}{\partial x} = 2x$ at $y = \dfrac{\pi}{2}$ $\qquad \therefore \quad \phi(x) = 2x - \sin x$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x - \sin x(1 - \sin y) \qquad \therefore \quad u = x^2 + \cos x(1 - \sin y) + \theta(y)$$

But $u = 2\sin y$ at $x = \pi \quad \therefore \quad \theta(y) = 1 - \pi^2 + \sin y$

$$u = x^2 + \cos x(1 - \sin y) + \sin y + 1 - \pi^2$$

On to the next frame

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Before we take a closer look at some of the more important partial differential equations occurring in branches of technology, let us recall the fact that if $u = u_1$, $u = u_2$, $u = u_3, \ldots$ are different solutions of a linear partial differential equation, so also is the *linear combination*

$$u = c_1u_1 + c_2u_2 + c_3u_3 + \ldots$$

where $c_1, c_2, c_3, \ldots$ are arbitrary constants.

There are many types of partial differential equations, some requiring special treatment in their solution. In this Programme we are concerned with a restricted number of such equations that occur in branches of science and technology, which can be solved by the method of separating the variables, and which also link up with the work we have done on Fourier series techniques.

Let us make a new start

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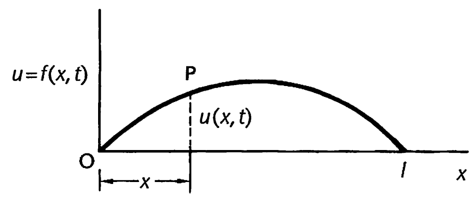
The wave equation



Consider a perfectly flexible elastic string stretched between two points at $x = 0$ and $x = l$ with uniform tension T. If the string is displaced slightly from its initial position of rest and released, with the end points remaining fixed, then the string will vibrate. The position of any point P in the string will then depend on its distance from one end and on the instant in time. Its displacement u at any time t can thus be expressed as $u = f(x, t)$ where x is its distance from the left-hand end.

The equation of motion is given by $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$, where $c^2 = \dfrac{T}{\rho}$ in which T is the tension in the string and ρ the mass per unit length of the string. The displacement of the string is regarded as small so that T and ρ remain constant.

Now let us deal with the solution of this equation.

On to the next frame

Solution of the wave equation

The new equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$ has a solution $u(x, t)$.

Boundary conditions:

(a) The string is fixed at both ends, i.e. at $x = 0$ and at $x = l$ for all values of time t. Therefore $u\,(x, t)$ becomes

$$\left.\begin{aligned} u(0, t) &= 0 \\ u(l, t) &= 0 \end{aligned}\right\} \quad \text{for all values of } t \geq 0$$

Initial conditions:

(b) If the initial deflection of P at $t = 0$ is denoted by $f(x)$, then

$$u(x, 0) = f(x)$$

(c) Let the initial velocity of P be $g(x)$, then

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x)$$

So now we have listed all the information available from the question. Next we turn to solving the equation.

Solution by separating the variables

We assume a trial solution of the form $u(x, t) = X(x)T(t)$ where

$X(x)$ is a function of x only
$T(t)$ is a function of t only.

If we simplify the symbols to $u = XT$ and denote derivatives with respect to their own independent variables by primes, we have

$$u = XT \quad \therefore \quad \frac{\partial u}{\partial x} = X'T \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

$$\frac{\partial u}{\partial t} = XT' \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = XT''$$

The wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$ can then be written as

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$$\boxed{X''T = \frac{1}{c^2}XT''}$$

and this can be transposed into $\dfrac{X''}{X} = \dfrac{1}{c^2} \cdot \dfrac{T''}{T}$

Notice that the left-hand side expression involves functions of x only and that the right-hand side expression involves functions of t only. Therefore, if these two expressions are to be equal for all values of the separate variables, then both expressions must be equal to

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$$\boxed{\text{a constant}}$$

Denote this arbitrary constant by k. Then we have

$$\frac{X''}{X} = k \quad \text{and} \quad \frac{1}{c^2} \cdot \frac{T''}{T} = k$$

$$\therefore\ X'' - kX = 0 \quad \text{and} \quad T'' - c^2kT = 0$$

Let us consider the first of these two equations for different values of k.

(1) *If* $k = 0$, $X'' = 0 \quad \therefore\ X' = a \quad \therefore\ X = ax + b$.

$$\left.\begin{array}{l}\text{But } X = 0 \text{ at } x = 0 \quad \therefore\ b = 0 \quad \therefore\ X = ax \\ \text{and } X = 0 \text{ at } x = l \quad \therefore\ a = 0\end{array}\right\} \therefore\ a = b = 0$$

$\therefore\ X = 0$ which is not oscillatory as the problem requires it to be.

(2) *If* k *is positive,* let $k = p^2 \quad \therefore\ X'' - p^2X = 0$.

The auxiliary equation is therefore $m^2 - p^2 = 0 \quad \therefore\ m^2 = p^2$

$$m = \pm p$$

$\therefore\ X = Ae^{px} + Be^{-px}$

But $X = 0$ at $x = 0 \quad \therefore\ 0 = A + B \quad \therefore\ B = -A$

and $X = 0$ at $x = l \quad \therefore\ 0 = Ae^{pl} - Ae^{-pl} \quad \therefore\ 0 = A(e^{pl} - e^{-pl})$

$$\therefore\ A = 0 \quad \therefore\ A = B = 0$$

Here again $X = 0$ which is not oscillatory.

(3) *If* k *is negative,* let $k = -p^2 \quad \therefore\ X'' + p^2X = 0$.

This is one of the standard equations listed at the beginning of the Programme and gives a solution

$$X = A\cos px + B\sin px \qquad (1)$$

which fits the requirements.

The second equation $T'' - c^2kT = 0$ therefore now becomes

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$$\boxed{T'' + c^2p^2T = 0}$$

because the same value for k must apply. This equation is of the same form as before and gives the solution

$$T = C\cos cpt + D\sin cpt \qquad (2)$$

So our suggested solution $u = XT$ now becomes

$$u(x, t) = (A\cos px + B\sin px)\,(C\cos cpt + D\sin cpt)$$

and, if we put $cp = \lambda \quad \therefore\ p = \dfrac{\lambda}{c}$, this becomes

$$u(x, t) = \left(A\cos\frac{\lambda}{c}x + B\sin\frac{\lambda}{c}x\right)(C\cos\lambda t + D\sin\lambda t) \qquad (3)$$

where A, B, C, D are arbitrary constants.

The result, of course, must satisfy the set of boundary conditions which we now turn to.

(a) $u = 0$ when $x = 0$ for all values of t. From this, we get

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$$\boxed{A = 0}$$

Because, substituting $u = 0$ and $x = 0$ in result (3) above

$$0 = A(C\cos\lambda t + D\sin\lambda t) \text{ for all } t \quad \therefore\ A = 0$$

$$\therefore\ u(x, t) = B\sin\frac{\lambda}{c}x(C\cos\lambda t + D\sin\lambda t)$$

(b) $u = 0$ when $x = l$ for all $t \quad \therefore\ 0 = B\sin\dfrac{\lambda l}{c}(C\cos\lambda t + D\sin\lambda t)$

Now $B \neq 0$ or $u(x, t)$ would be identically zero. $\therefore\ \sin\dfrac{\lambda l}{c} = 0$.

$$\therefore\ \frac{\lambda l}{c} = n\pi \text{ where } n = 1, 2, 3, \ldots \quad \therefore\ \lambda = \frac{nc\pi}{l} \text{ for } n = 1, 2, 3, \ldots$$

Note that we exclude $n = 0$ since this would also make $u(x, t)$ identically zero.

▶

As we can see, there is an infinite set of values of λ and each separate value gives a particular solution for $u(x,t)$. The values of λ are called the *eigenvalues* and each corresponding solution the *eigenfunction*.

Putting $n = 1,\ 2,\ 3, \dots$ we therefore have

	Eigenvalues	Eigenfunctions
n	$\lambda = \frac{nc\pi}{l}$	$u(x,t) = B\sin\frac{\lambda x}{c}\{C\cos\lambda t + D\sin\lambda t\}$
1	$\lambda_1 = \frac{c\pi}{l}$	$u_1 = \sin\frac{\pi x}{l}\left\{C_1\cos\frac{c\pi t}{l} + D_1\sin\frac{c\pi t}{l}\right\}$
2	$\lambda_2 = \frac{2c\pi}{l}$	$u_2 = \sin\frac{2\pi x}{l}\left\{C_2\cos\frac{2c\pi t}{l} + D_2\sin\frac{2c\pi t}{l}\right\}$
3	$\lambda_3 = \frac{3c\pi}{l}$	$u_3 = \sin\frac{3\pi x}{l}\left\{C_3\cos\frac{3c\pi t}{l} + D_3\sin\frac{3c\pi t}{l}\right\}$
$\vdots$	$\vdots$	$\vdots$
r	$\lambda_r = \frac{rc\pi}{l}$	$u_r = \sin\frac{r\pi x}{l}\left\{C_r\cos\frac{rc\pi t}{l} + D_r\sin\frac{rc\pi t}{l}\right\}$

Note that the constant B has been absorbed into the constants C and D so that $BC = C_n$ and $BD = D_n$, where $C_1,\ C_2,\ C_3, \dots$ and $D_1,\ D_2,\ D_3, \dots$ are arbitrary constants.

Since the original wave equation is linear in form, we have already noted that if $u = u_1,\ u = u_2,\ u = u_3 \dots$ are particular solutions, a more general solution is

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$$\boxed{u = u_1 + u_2 + u_3 + \dots}$$

The more general solution is therefore

$$u(x,t) = \sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty}\left\{\sin\frac{r\pi x}{l}\left(C_r\cos\frac{rc\pi t}{l} + D_r\sin\frac{rc\pi t}{l}\right)\right\} \tag{4}$$

We still have to find C_r and D_r and for this we use the initial conditions which we have not yet taken into account.

(c) At $t = 0$, $u(x,\ 0) = f(x)$ for $0 \le x \le l$

Therefore from (4), $u(x,\ 0) = f(x) = \sum_{r=1}^{\infty} C_r\ \sin\frac{r\pi x}{l}$.

(d) Also at $t = 0$, $\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x)$ for $0 \le x \le l$

We therefore differentiate (4) with respect to t and put $t = 0$, which gives

....

$$\boxed{g(x) = \frac{c\pi}{l}\sum_{r=1}^{\infty} D_r r \sin\frac{r\pi x}{l}}$$

Because

$$\frac{\partial u}{\partial t} = \sum_{r=1}^{\infty} \sin\frac{r\pi x}{l}\left\{-C_r\frac{rc\pi}{l}\sin\frac{rc\pi t}{l} + D_r\frac{rc\pi}{l}\cos\frac{rc\pi t}{l}\right\}$$

$$\therefore \text{ With } t = 0, \quad \frac{\partial u}{\partial t} = g(x) = \sum_{r=1}^{\infty} D_r\frac{rc\pi}{l}\sin\frac{r\pi x}{l}$$

$$\therefore\ g(x) = \frac{c\pi}{l}\sum_{r=1}^{\infty} D_r r \sin\frac{r\pi x}{l}$$

Finally we can draw on our knowledge of Fourier series techniques to determine the coefficients C_r and D_r.

$C_r = 2 \times$ mean value of $f(x)\sin\dfrac{r\pi x}{l}$ between $x = 0$ and $x = l$

$$\therefore\ C_r = \frac{2}{l}\int_0^l f(x)\sin\frac{r\pi x}{l}\mathrm{d}x \qquad r = 1, 2, 3, \ldots$$

and $D_r\dfrac{rc\pi}{l} = 2 \times$ mean value of $g(x)\sin\dfrac{r\pi x}{l}$ between $x = 0$ and $x = l$

$$\therefore\ D_r = \frac{2}{rc\pi}\int_0^l g(x)\sin\frac{r\pi x}{l}\mathrm{d}x \qquad r = 1, 2, 3, \ldots$$

The general solution (4) then becomes

$$u(x, t) = \sum_{r=1}^{\infty}\left\{\left[\frac{2}{l}\int_0^l f(w)\sin\frac{r\pi w}{l}\mathrm{d}w\right]\cos\frac{rc\pi t}{l}\sin\frac{r\pi x}{l}\right.$$
$$\left. + \left[\frac{2}{rc\pi}\int_0^l g(w)\sin\frac{r\pi w}{l}\mathrm{d}w\right]\sin\frac{rc\pi t}{l}\sin\frac{r\pi x}{l}\right\} \qquad (5)$$

Notice that the variable of integration has been changed from x to w because we wish to use the variable x in the final expression for $u(x, t)$. The value of a definite integral depends only on the limit points of the integral and we are free to use any symbol that we desire for the variable of integration – we call such a variable a *dummy variable*.

▶

At first sight, the solution seems very involved, but it can be analysed into a definite sequence of logical steps. Given the equation and relevant initial and boundary conditions, we go through the following stages.

(a) Assume a solution of the form $u = XT$ and express the equation in terms of X and T and their derivatives.

(b) Transpose the equation by separation of the variables and equate each side to a constant, so obtaining two separate equations, one in x and the other in t.

(c) Choose $k = -p^2$ to give an oscillatory solution.

(d) The two solutions are of the form

$$X = A\cos px + B\sin px$$
$$T = C\cos cpt + D\sin cpt$$

Then $u(x,t) = \{A\cos px + B\sin px\}\{C\cos cpt + D\sin cpt\}$.

(e) Putting $cp = \lambda$, i.e. $p = \frac{\lambda}{c}$, this becomes

$$u(x,t) = \left\{A\cos\frac{\lambda}{c}x + B\sin\frac{\lambda}{c}x\right\}\{C\cos\lambda t + D\sin\lambda t\}.$$

(f) Apply boundary conditions to determine A and B.

(g) List the eigenvalues and eigenfunctions for $n = 1, 2, 3, \ldots$ and determine the general solution as an infinite sum.

(h) Apply the remaining initial or boundary conditions.

(i) Determine the coefficients C_r and D_r by Fourier series techniques.

Make a list of these steps: then we can follow them with an example.

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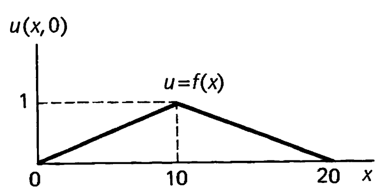
Example



A stretched string of length 20 cm is set oscillating by displacing its mid-point a distance 1 cm from its rest position and releasing it with zero initial velocity. Solve the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\cdot\frac{\partial^2 u}{\partial t^2}$ where $c^2 = 1$ to determine the resulting motion, $u(x,t)$.

First we make a list of the boundary conditions from the data given in the question.

$u(0,\ t) = \ldots\ldots\ldots\ldots$; $\quad u(20,\ t) = \ldots\ldots\ldots\ldots$

$u(x,\ 0) = \ldots\ldots\ldots\ldots$

$\qquad\qquad\ \ldots\ldots\ldots\ldots$

$\left[\frac{\partial u}{\partial t}\right]_{t=0} = \ldots\ldots\ldots\ldots$

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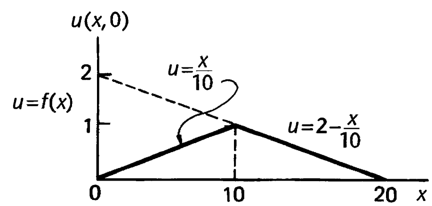
$$\boxed{\begin{aligned} &u(0,t)=0; \quad u(20,t)=0 \quad \text{(fixed end points)} \\ &u(x,0)=f(x)=\frac{x}{10} \qquad 0 \le x \le 10 \\ &\qquad\qquad\quad =\frac{20-x}{10} \quad 10 \le x \le 20 \\ &\left[\frac{\partial u}{\partial t}\right]_{t=0}=0 \quad \text{(zero initial velocity)} \end{aligned}}$$



Now we can apply our sequence of operations which we listed.

So move on

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(a) Assume a solution $u = XT$ where X is a function of x only and T is a function of t only. Then the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ (since $c = 1$) becomes ....

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$$\boxed{X''T = XT''}$$

Because

$$u = XT \qquad \therefore \quad \frac{\partial u}{\partial x} = X'T \qquad \frac{\partial^2 u}{\partial x^2} = X''T$$

$$\text{and} \qquad \frac{\partial u}{\partial t} = XT' \qquad \frac{\partial^2 u}{\partial t^2} = XT''$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \therefore \quad X''T = XT''$$

(b) Next we rearrange the equation to separate the variables, giving

....

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$$\boxed{\frac{X''}{X} = \frac{T''}{T}}$$

(c) Since the two sides are equal for all values of the variables, each must be equal to a constant k and to give an oscillatory solution we put $k = -p^2$. The two separate equations then are written

.... and ....

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$$X'' + p^2X = 0 \quad \text{and} \quad T'' + p^2T = 0$$

(d) These have solution $X = \dots\dots\dots\dots$
$T = \dots\dots\dots\dots$
so that $u(x, t) = \dots\dots\dots\dots$

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$$X = A\cos px + B\sin px; \quad T = C\cos pt + D\sin pt$$
$$\therefore \; u(x, t) = \{A\cos px + B\sin px\}\{C\cos pt + D\sin pt\}$$

(e) We normally now put $cp = \lambda$, but in this case $c = 1 \;\therefore\; p = \lambda$ and $u(x, t) = \dots\dots\dots\dots$

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$$u(x, t) = \{A\cos\lambda x + B\sin\lambda x\}\{C\cos\lambda t + D\sin\lambda t\}$$

(f) Now we determine A and B from the boundary conditions.

(1) $u(0, t) = 0 \quad \therefore\; 0 = A(C\cos\lambda t + D\sin\lambda t) \quad \therefore\; A = 0$
$\therefore\; u(x, t) = B\sin\lambda x(C\cos\lambda t + D\sin\lambda t)$

(2) $u(20, t) = 0 \quad \therefore\; 0 = B\sin 20\lambda(C\cos\lambda t + D\sin\lambda t)$
$B \neq 0$ or u would be identically zero. $\quad \therefore\; \sin 20\lambda = 0$.
$\therefore\; 20\lambda = n\pi \quad \therefore\; \lambda = \dfrac{n\pi}{20}$
$\therefore\; u(x, t) = \sin\dfrac{n\pi}{20}x\left\{P\cos\dfrac{n\pi}{20}t + Q\sin\dfrac{n\pi}{20}t\right\}$
where $P = B \times C$ and $Q = B \times D$.

(g) The next step is to list the eigenvalues and eigenfunctions.

	Eigenvalues	Eigenfunctions
n	$\lambda = \dfrac{n\pi}{20}$	$u(x,t) = \sin\lambda x\{P\cos\lambda t + Q\sin\lambda t\}$
1	$\lambda_1 = \dfrac{\pi}{20}$	$u_1 = \sin\dfrac{\pi x}{20}\left\{P_1\cos\dfrac{\pi t}{20} + Q_1\sin\dfrac{\pi t}{20}\right\}$
2	$\lambda_2 = \dfrac{2\pi}{20}$	$u_2 = \sin\dfrac{2\pi x}{20}\left\{P_2\cos\dfrac{2\pi t}{20} + Q_2\sin\dfrac{2\pi t}{20}\right\}$
3	$\lambda_3 = \dfrac{3\pi}{20}$	$u_3 = \sin\dfrac{3\pi x}{20}\left\{P_3\cos\dfrac{3\pi t}{20} + Q_3\sin\dfrac{3\pi t}{20}\right\}$
$\vdots$	$\vdots$	$\vdots$
r	$\lambda_r = \dfrac{r\pi}{20}$	$u_r = \sin\dfrac{r\pi x}{20}\left\{P_r\cos\dfrac{r\pi t}{20} + Q_r\sin\dfrac{r\pi t}{20}\right\}$

$$u = u_1 + u_2 + u_3 + \dots \quad \therefore\; u(x, t) = \sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}\left\{P_r\cos\frac{r\pi t}{20} + Q_r\sin\frac{r\pi t}{20}\right\}$$

▶

(h) Now we apply the remaining initial conditions

$$(1)\quad u(x,0) = f(x) = \frac{x}{10} \qquad 0 \le x \le 10$$
$$= \frac{20-x}{10} \qquad 10 \le x \le 20$$

Also $u(x,0) = \dots\dots\dots\dots$

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$$\boxed{u(x,\,0) = \sum_{r=1}^{\infty} P_r \sin\frac{r\pi x}{20}}$$

Then $P_r = 2 \times$ mean value of $f(x)\,\sin\dfrac{r\pi x}{20}$ between $x = 0$ and $x = 20$

$$= \frac{2}{20}\int_0^{20} f(x)\,\sin\frac{r\pi x}{20}\mathrm{d}x$$

$$\therefore\ 10P_r = \int_0^{10}\frac{x}{10}\,\sin\frac{r\pi x}{20}\mathrm{d}x + \int_{10}^{20}\frac{20-x}{10}\sin\frac{r\pi x}{20}\mathrm{d}x$$
$$= \qquad I_1 \qquad + \qquad I_2$$

$$I_1 = \int_0^{10}\frac{x}{10}\sin\frac{r\pi x}{20}\mathrm{d}x = \dots\dots\dots\dots$$

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$$\boxed{I_1 = -\frac{20}{r\pi}\cos\frac{r\pi}{2} + \frac{40}{r^2\pi^2}\sin\frac{r\pi}{2}}$$

Using integration by parts

$$I_2 = \int_{10}^{20}\frac{20-x}{10}\,\sin\frac{r\pi x}{20}\,\mathrm{d}x = \dots\dots\dots\dots$$

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$$\boxed{I_2 = \frac{20}{r\pi}\cos\frac{r\pi}{2} - \frac{40}{r^2\pi^2}\left(\sin r\pi - \sin\frac{r\pi}{2}\right)}$$

$$\text{Then } 10P_r = -\frac{20}{r\pi}\cos\frac{r\pi}{2} + \frac{40}{r^2\pi^2}\sin\frac{r\pi}{2} + \frac{20}{r\pi}\cos\frac{r\pi}{2} - \frac{40}{r^2\pi^2}\left(\sin r\pi - \sin\frac{r\pi}{2}\right)$$

$$\therefore\ \text{For } r = 1,\,2,\,3,\dots\ P_r = \frac{8}{r^2\pi^2}\sin\frac{r\pi}{2}$$

$$\therefore\ u(x,t) = \sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}\left\{\frac{8}{r^2\pi^2}\sin\frac{r\pi}{2}\cos\frac{r\pi t}{20} + Q_r\sin\frac{r\pi t}{20}\right\}$$

(2) Also at $t = 0$, $\dfrac{\partial u}{\partial t} = 0.$

$$\frac{\partial u}{\partial t} = \dots\dots\dots\dots$$

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$$\frac{\partial u(x,t)}{\partial t}=\sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}\left\{\left(\frac{8}{r^2\pi^2}\sin\frac{r\pi}{2}\right)\left(-\frac{r\pi}{20}\sin\frac{r\pi t}{20}\right)+Q_r\frac{r\pi}{20}\cos\frac{r\pi t}{20}\right\}$$

$\therefore$ At $t=0$, $\quad 0=\sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}Q_r\frac{r\pi}{20} \quad \therefore\ Q_r=0$

So finally we have $\quad u(x,t)=\dots\dots\dots\dots$

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$$u(x,t)=\frac{8}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}\sin\frac{r\pi x}{20}\sin\frac{r\pi}{2}\cos\frac{r\pi t}{20}$$

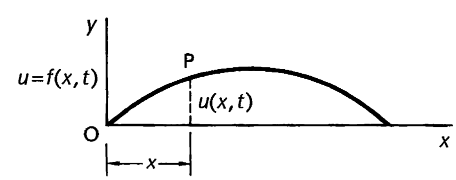
And that is it.

Now let us turn to a slightly different equation, but one for which the method of solution is very much along the same lines.

The heat conduction equation for a uniform finite bar

The conduction of heat in a uniform bar depends on the initial distribution of temperature and on the physical properties of the bar, i.e. the thermal conductivity and specific heat of the material, and the mass per unit length of the bar.

With a uniform bar insulated except at its ends, any heat flow is along the bar and, at any instant, the temperature u at a point P is a function of its distance x from one end and of the time t.



The one-dimensional heat equation is then of the form

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\cdot\frac{\partial u}{\partial t} \tag{1}$$

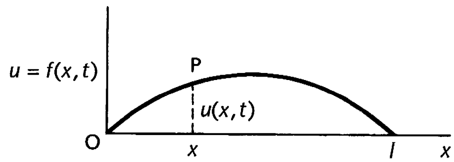
where $c^2=\frac{k}{\sigma\rho}$ in which $k=$ thermal conductivity of the material; $\sigma=$ specific heat of the material; $\rho=$ mass per unit length of the bar.

You will already have noticed that the heat equation differs from the wave equation only in the fact that the right-hand side contains the first partial derivative instead of the second. It is not surprising therefore that the method of solution is very much like that of our previous examples.

Solutions of the heat conduction equation

Consider the case where

(a) the bar extends from $x = 0$ to $x = l$
(b) the temperature of the ends of the bar is maintained at zero
(c) the initial temperature distribution along the bar is defined by $f(x)$.



The boundary conditions can be expressed as ....

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$$\boxed{\begin{array}{l} u(0, t) = 0 \quad \text{and} \quad u(l, t) = 0 \quad \text{for all } t \geq 0 \\ u(x, 0) = f(x) \quad \text{for} \quad 0 \leq x \leq l \end{array}}$$

As before, we assume a solution of the form $u(x, t) = X(x)T(t)$ where

X is a function of x only
T is a function of t only.

Then, starting with $u = XT$ we can write the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial u}{\partial t}$ in terms of X and T, and separating the variables, we obtain

....

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$$\boxed{\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T'}{T}}$$

Arguing as before, since the left-hand side is a function of x only and the right-hand side a function of t only, for these to be equal each side must equal the same constant. Let this be $(-p^2)$ as before.

$\therefore \; \dfrac{X''}{X} = -p^2 \quad \therefore \; X'' + p^2 X = 0$ giving $X = A\cos px + B\sin px$

and $\dfrac{1}{c^2} \cdot \dfrac{T'}{T} = -p^2 \quad \therefore \; T' + p^2c^2T = 0$ giving $T = ...$

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$$\boxed{T = Ce^{-p^2c^2t}}$$

Because

$$\frac{T'}{T} = -p^2c^2 \quad \therefore \quad \ln T = -p^2c^2t + c_1 \quad \therefore \quad T = Ce^{-p^2c^2t}$$

$$u(x,t) = XT = \{A\cos px + B\sin px\}\,Ce^{-p^2c^2t}$$

$$\therefore \quad u(x,t) = \{P\cos px + Q\sin px\}e^{-p^2c^2t} \qquad \text{where } P = AC \text{ and } Q = BC$$

Now put $pc = \lambda \quad \therefore \quad p = \dfrac{\lambda}{c}$

$$\therefore \quad u(x,t) = \left\{P\cos\frac{\lambda}{c}x + Q\sin\frac{\lambda}{c}x\right\}e^{-\lambda^2 t}$$

Applying the boundary condition $u(0,t) = 0$ gives

.... and ....

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$$\boxed{P = 0 \ \text{ and } \ u(x,t) = Qe^{-\lambda^2 t}\sin\frac{\lambda}{c}x}$$

Also $u(l,t) = 0$ and from this we get

....

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$$\boxed{\lambda = \frac{nc\pi}{l} \quad \text{for } \ n = 1,\ 2,\ 3, \ldots}$$

Because

if $u = 0$ when $x = l$, $\qquad 0 = Qe^{-\lambda^2 t}\sin\dfrac{\lambda l}{c}$

$Q \neq 0$ or $u(x,\,t)$ would be identically zero $\therefore$ $\sin\dfrac{\lambda l}{c} = 0$

$$\therefore \quad \frac{\lambda l}{c} = n\pi \qquad \therefore \quad \lambda = \frac{nc\pi}{l} \qquad n = 1,\ 2,\ 3, \ldots$$

▶

Now we can compile the table of eigenfunctions.

n	$\lambda = \dfrac{nc\pi}{l}$	$u(x,t) = Qe^{-\lambda^2 t}\sin\dfrac{n\pi x}{l}$
1	$\lambda_1 = \dfrac{c\pi}{l}$	$u_1 = Q_1 e^{-\lambda_1^2 t}\sin\dfrac{\pi x}{l}$
2	$\lambda_2 = \dfrac{2c\pi}{l}$	$u_2 = Q_2 e^{-\lambda_2^2 t}\sin\dfrac{2\pi x}{l}$
3	$\lambda_3 = \dfrac{3c\pi}{l}$	$u_3 = Q_3\, e^{-\lambda_3^2 t}\sin\dfrac{3\pi x}{l}$
$\vdots$	$\vdots$	$\vdots$
r	$\lambda_r = \dfrac{rc\pi}{l}$	$u_r = Q_r\, e^{-\lambda_r^2 t}\sin\dfrac{r\pi x}{l}$

$$u = u_1 + u_2 + u_3 + \dots$$

$$\therefore\; u(x,\,t) = \sum_{r=1}^{\infty}\left\{Q_r\, e^{-\lambda_r^2 t}\sin\frac{r\pi x}{l}\right\}$$

The remaining boundary condition still to be applied is that when

$t = 0,\;\; u(x,0) = f(x) \quad 0 \leq x \leq l$

This gives $f(x) = \dots\dots\dots\dots$

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$$\boxed{f(x) = \sum_{r=1}^{\infty}\left\{Q_r \sin\frac{r\pi x}{l}\right\}}$$

and from our knowledge of Fourier series techniques:

$Q_r = \dots\dots\dots\dots$

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$$\boxed{Q_r = 2 \times \text{mean value of } f(x)\sin\frac{r\pi x}{l} \text{ from } x = 0 \text{ to } x = l}$$

$\therefore\; Q_r = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{r\pi x}{l}\,\mathrm{d}x$ and the final solution becomes

$$u(x,t) = \frac{2}{l}\sum_{r=1}^{\infty}\left\{\left[\int_0^l f(w)\sin\frac{r\pi w}{l}\,\mathrm{d}w\right] e^{-\lambda_r^2 t}\sin\frac{r\pi x}{l}\right\}$$

where $\lambda_r = \dfrac{rc\pi}{l} \qquad r = 1,\, 2,\, 3,\, \dots$

Now on to the next frame for an example

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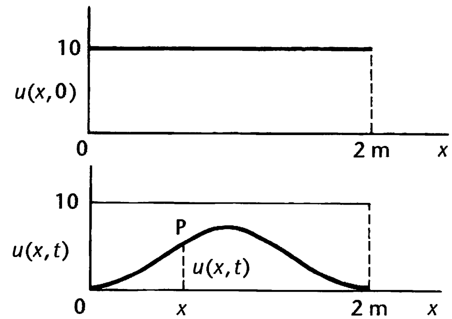
Example

A bar of length 2 m is fully insulated along its sides. It is initially at a uniform temperature of 10°C and at $t = 0$ the ends are plunged into ice and maintained at a temperature of 0°C. Determine an expression for the temperature at a point P a distance x from one end at any subsequent time t seconds after $t = 0$.



We have the heat equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial u}{\partial t}$ with the boundary conditions

............; ....; and ....

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$$\boxed{u(0, t) = 0; \quad u(2, t) = 0; \quad u(x, 0) = 10}$$

Assuming a solution of the form $u = XT$, we know that this gives for this equation $\quad X = A\cos px + B\sin px$

and $\quad T = Ce^{-p^2c^2t}$

so that the general solution is

$$u(x, t) = \{P\cos px + Q\sin px\}e^{-p^2c^2t}$$

If we now write $pc = \lambda$, $\quad p = \dfrac{\lambda}{c}$ and the solution becomes

$$u(x, t) = \left\{P\cos\frac{\lambda}{c}x + Q\sin\frac{\lambda}{c}x\right\}e^{-\lambda^2 t}$$

Applying the first two of the boundary conditions gives us

....

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$$\boxed{P = 0 \quad \text{and} \quad u(x, t) = \left\{Q \sin\frac{n\pi x}{2}\right\}e^{-\lambda^2 t}}$$

Because

$u(0, t) = 0 \quad \therefore \; 0 = Pe^{-\lambda^2 t} \quad \therefore \; P = 0$

$$\therefore \; u(x, t) = \left\{Q \sin\frac{\lambda}{c}x\right\}e^{-\lambda^2 t}$$

Also $u(2, t) = 0 \qquad \therefore \; 0 = \left\{Q \sin\frac{2\lambda}{c}\right\}e^{-\lambda^2 t}$

$Q \neq 0 \qquad \therefore \; \sin\frac{2\lambda}{c} = 0 \quad \therefore \; \frac{2\lambda}{c} = n\pi \quad \therefore \; \lambda = \frac{nc\pi}{2} \qquad n = 1, 2, 3, \ldots$

$$\therefore \; u(x, t) = \left\{Q \sin\frac{n\pi x}{2}\right\}e^{-\lambda^2 t}$$

There is, of course, an infinite number of such solutions with different values of n. We can write the solution so far therefore as

$$u(x, t) = \ldots\ldots\ldots\ldots$$

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$$\boxed{u(x, t) = \sum_{r=1}^{\infty} Q_r \sin\frac{r\pi x}{2} e^{-\lambda_r^2 t}}$$

Finally, there is the remaining initial condition that at $t = 0$, $u = 10$.

$$\therefore \; u(x, 0) = f(x) = 10 \quad \therefore \; 10 = \sum_{r=1}^{\infty} Q_r \sin\frac{r\pi x}{2}$$

where $Q_r = 2 \times$ mean value of $10 \sin\frac{r\pi x}{2}$ from $x = 0$ to $x = 2$.

$$\therefore \; Q_r = \ldots\ldots\ldots\ldots$$

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$$\boxed{0 \; (r \text{ even}); \quad \frac{40}{\pi r} \; (r \text{ odd})}$$

Because

$$Q_r = \frac{2}{2}\int_0^2 10 \sin\frac{r\pi x}{2}\,dx = 10\int_0^2 \sin\frac{r\pi x}{2}\,dx$$

$$= -\frac{20}{\pi r}\left[\cos\frac{r\pi x}{2}\right]_0^2 = \frac{20}{\pi r}\{1 - \cos r\pi\}$$

$$= 0 \; (r \text{ even}) \text{ and } \frac{40}{r\pi} \; (r \text{ odd})$$

Therefore the required solution is

$$u(x, t) = \ldots\ldots\ldots\ldots$$

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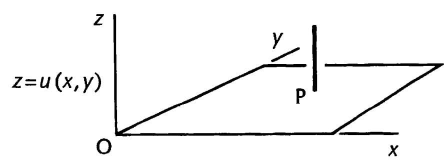
$$u(x, t) = \frac{40}{\pi} \sum_{r\,(\text{odd})=1}^{\infty} \frac{1}{r} \sin\frac{r\pi x}{2} e^{-\lambda_r^2 t} \qquad r = 1, 3, 5, \ldots$$

where $\lambda_r = \dfrac{rc\pi}{2}$

By now you will appreciate that the approach to all these problems is very much the same, as indeed it still is with the next important equation.

Laplace's equation

The Laplace equation concerns the distribution of a field, e.g. temperature, potential, etc., over a plane area subject to certain boundary conditions.



The potential at a point P in a plane can be indicated by an ordinate axis and is a function of its position, i.e. $z = u(x, y)$ where $u(x, y)$ is the solution of the Laplace two-dimensional equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

Let us consider the situation in the next frame

Solution of the Laplace equation

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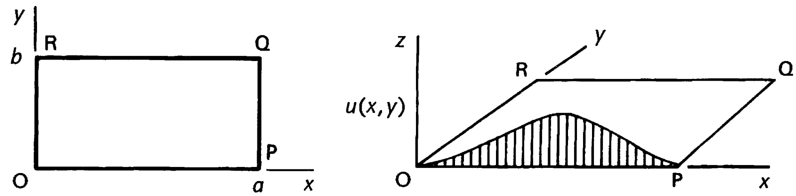

We are required to determine a solution of the equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ for the rectangle bounded by the lines $x=0$, $y=0$, $x=a$, $y=b$, subject to the following boundary conditions

$$\begin{array}{llll} u=0 & \text{when} & x=0 & 0\le y\le b \\ u=0 & \text{when} & x=a & 0\le y\le b \\ u=0 & \text{when} & y=b & 0\le x\le a \\ u=f(x) & \text{when} & y=0 & 0\le x\le a \end{array}$$

i.e. $u(0, y)=0$ and $u(a, y)=0$ for $0\le y\le b$
and $u(x, b)=0$ and $u(x, 0)=f(x)$ for $0\le x\le a$.

The solution $z=u(x, y)$ will give the potential at any point within the rectangle OPQR.

We start off, as usual, by assuming a solution of the form $u(x, y)=X(x)Y(y)$ where X is a function of x only and Y is a function of y only. We now express the equation in terms of X and Y and separate the variables to give

. . . .

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$$\boxed{\frac{X''}{X}=-\frac{Y''}{Y}}$$

Because

$$u=XY \qquad \therefore\ \frac{\partial u}{\partial x}=X'Y \quad\text{and}\quad \frac{\partial^2 u}{\partial x^2}=X''Y$$
$$\frac{\partial u}{\partial y}=XY' \quad\text{and}\quad \frac{\partial^2 u}{\partial y^2}=XY''$$

The equation is then $X''Y=-XY''$ $\quad\therefore\ \dfrac{X''}{X}=-\dfrac{Y''}{Y}$

Putting each side equal to a constant $(-p^2)$ gives two equations

$X''+p^2X=0$ and $Y''-p^2Y=0$

$X''+p^2X=0$ has a solution $X=$ . . . .

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$$\boxed{X = A\cos px + B\sin px}$$

In the introduction to this Programme we said that the equation $Y'' - p^2 Y = 0$ has a solution of the form $Y = C\cosh py + D\sinh py$ which can also be expressed as $Y = E\sinh p(y + \phi)$.

$$\therefore\ u(x, y) = \{A\cos px + B\sin px\}E\sinh p(y + \phi)$$

$$\therefore\ u(x, y) = \{P\cos px + Q\sin px\}\sinh p(y + \phi)$$

Now we apply the first of the boundary conditions.

$$u(0, y) = 0 \quad \therefore\ 0 = P\sinh p(y + \phi) \quad \therefore\ P = 0$$

$$\therefore\ u(x, y) = Q\sin px \sinh p(y + \phi)$$

From the second boundary condition, we have

$$u(a, y) = 0 \quad \therefore\ 0 = Q\sin pa \sinh p(y + \phi) \quad \therefore\ \sin pa = 0$$

$$\therefore\ pa = n\pi \qquad \text{for } n = 1, 2, 3, \dots$$

If we write $\lambda = p$ then $\lambda = \dfrac{n\pi}{a}$ and $u(x, y) = Q\sin\lambda x \sinh\lambda(y + \phi)$

Now from the third condition

$u(x, b) = 0$ from which we have ....

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$$\boxed{u(x, y) = Q\sin\lambda x \sinh\lambda(b - y)}$$

Because

$$0 = Q\sin\lambda x \sinh\lambda(b + \phi) \quad \therefore\ \sinh\lambda(b + \phi) = 0 \quad \therefore\ \phi = -b.$$

$$\therefore\ u(x, y) = Q\sin\lambda x \sinh\lambda(y - b)$$

$$\sinh\lambda(y - b) = -\sinh\lambda(b - y) \quad \therefore\ u(x, y) = Q\sin\lambda x \sinh\lambda(b - y),$$

the minus sign being absorbed in the symbol Q whose value has yet to be found. Now $\lambda = \dfrac{n\pi}{a}$ with $n = 1, 2, 3, \dots$ and there is therefore an infinite number of values for λ and hence an infinite number of solutions for $u(x, y)$. Therefore, again using $u = u_1 + u_2 + u_3 + \dots$ we have $u(x, y) = $ ....

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$$\boxed{u(x, y) = \sum_{r=1}^{\infty} Q_r \sin \lambda_r x \sinh \lambda_r(b - y)}$$

Now there remains the fourth boundary condition to be applied.

$$u(x, 0) = f(x) \quad \therefore \; f(x) = \sum_{r=1}^{\infty} Q_r \sin \lambda_r x \, \sinh \lambda_r b$$

$$\therefore \; Q_r \sinh \lambda_r b = 2 \times \text{ mean value of } f(x) \sin \lambda_r x \text{ from } x = 0 \text{ to } x = a$$

$$= \frac{2}{a} \int_0^a f(x) \sin \lambda_r x \, dx$$

$$= \frac{2}{a} \int_0^a f(x) \sin \frac{r\pi x}{a} \, dx$$

from which the coefficients Q_r can be found.

Let us work through an example with numerical values.

Example

Determine a solution $u(x, y)$ of the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ subject to the following boundary conditions

$u = 0$ when $x = 0$; $\quad u = 0$ when $x = \pi$

$u \to 0$ when $y \to \infty$; $\quad u = 3$ when $y = 0$

As always, we begin with $u(x, y) = X(x)Y(y)$, rewrite the equation in terms of X and Y and separate the variables. The equation then becomes

....

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$$\boxed{\frac{X''}{X} = -\frac{Y''}{Y}}$$

Equating each side to $-p^2$, we have $X'' + p^2 X = 0$ and $Y'' - p^2 Y = 0$.

$X'' + p^2 X = 0$ has a solution ....

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$$\boxed{X = A \cos px + B \sin px}$$

The solution of $Y'' - p^2 Y = 0$ can be stated in three different forms

$Y = C \cosh py + D \sinh py$; $\quad Y = Ce^{py} + De^{-py}$; $\quad Y = C \sinh p(y + \phi)$

On this occasion, we will use the second one

$$Y = Ce^{py} + De^{-py}$$

Then $u(x, y) = \{A \cos px + B \sin px\}\{Ce^{py} + De^{-py}\}$

Application of the first boundary condition $u(0, y) = 0$ gives

.... and ....

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$$\boxed{A = 0 \text{ and } u(x, y) = \sin px\{Pe^{py} + Qe^{-py}\}}$$

Because

$0 = A\{Ce^{py} + De^{-py}\} \quad \therefore A = 0$

and $u(x, y) = B\sin px\{Ce^{py} + De^{-py}\} = \sin px\{Pe^{py} + Qe^{-py}\}$.

The second boundary condition $u(\pi, y) = 0$ then gives

....

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$$\boxed{u(x, y) = \sin nx\,\{Pe^{ny} + Qe^{-ny}\} \qquad n = 1, 2, 3, \ldots}$$

Because

$u = 0$ when $x = \pi \quad \therefore \quad 0 = \sin p\pi\{Pe^{py} + Qe^{-py}\}$

$\therefore \quad \sin p\pi = 0 \quad \therefore \quad p\pi = n\pi \quad \therefore \quad p = n \qquad n = 1, 2, 3, \ldots$

$\therefore \quad u(x, y) = \sin\ nx\ \{Pe^{ny} + Qe^{-ny}\}$

The third condition is that $u \to 0$ as $y \to \infty$.

Because $e^{-ny} \to 0$ as $y \to \infty$ then $0 = \sin nx\{Pe^{ny}\}$, so that $P = 0$

$\therefore \quad u(x, y) = Qe^{-ny}\sin nx$

But n can have an infinite number of values giving an infinite number of solutions

$$\begin{aligned} u_1 &= Q_1e^{-y}\sin x \\ u_2 &= Q_2e^{-2y}\sin 2x \\ u_3 &= Q_3e^{-3y}\sin 3x \\ &\vdots \\ u_r &= Q_re^{-ry}\sin rx \end{aligned}$$

So the solution at this stage can be written as

$u(x, y) =$ ....

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$$\boxed{u(x, y) = \sum_{r=1}^{\infty} Q_re^{-ry}\sin rx}$$

Now we turn to the final boundary condition that $u = 3$ when $y = 0$.

$\therefore \quad 3 = \sum_{r=1}^{\infty} Q_r\sin rx$ from which we obtain

$Q_r =$ ....

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$$\boxed{Q_r = 0\ (r\text{ even});\quad Q_r = \frac{12}{r\pi}\ (r\text{ odd})}$$

Because

$$Q_r = 2\ \times\ \text{mean value of } 3\sin rx \text{ between } x = 0 \text{ and } x = \pi$$

$$= \frac{2}{\pi}\int_0^{\pi} 3\sin rx\ dx = \frac{6}{\pi}\left[-\frac{\cos rx}{r}\right]_0^{\pi} = \frac{6}{r\pi}(1 - \cos r\pi)$$

$$\therefore\ Q_r = 0\ \ (r\text{ even})\text{ and } \frac{12}{r\pi}\ \ (r\text{ odd})$$

$$\therefore\ u(x, y) = \sum_{r\,(\text{odd})=1}^{\infty} \frac{12}{r\pi} e^{-ry}\sin rx \qquad r = 1, 3, 5, \ldots$$

$$\therefore\ u(x, y) = \frac{12}{\pi}\left\{e^{-y}\sin x + \frac{1}{3}e^{-3y}\sin 3x + \frac{1}{5}e^{-5y}\sin 5x + \ldots\right\}$$

Laplace's equation in plane polar coordinates

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Laplace's equation

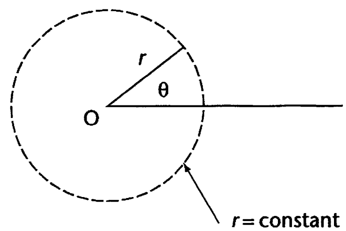
$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0$$

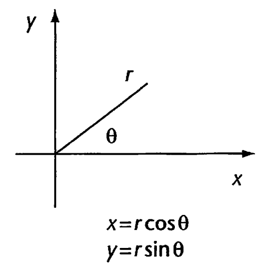
is often referred to as the *potential equation* because such physical entities as the electrostatic and gravitational potentials can be shown to satisfy it. It is an equation that is commonly met in science and engineering. Solving this equation inside a region of the x–y plane subject to some specified condition applied to $u(x, y)$ on the boundary of the region is known as a *Dirichlet problem*. To solve this Dirichlet problem we proceed, as we have seen, by separating the variables to find the general solution and then matching up the general solution to the boundary conditions to find the specific solution. However, the process of finding the specific solution from the general solution is very sensitive to the shape of the boundary, and difficulties can arise if the symmetries of the boundary do not match the symmetries of the coordinate system used. For example, if the region under consideration is bounded by the circle

$$x^2 + y^2 = a^2$$

employing Cartesian coordinates will create difficulties when we come to match up the general solution in Cartesians to the boundary conditions on the circular boundary. To avoid such difficulties we choose a coordinate system that has the same symmetries as the ▶

boundary where the coordinate symmetries are exhibited when we let one variable vary while keeping all the others constant. The Cartesian coordinate system (x, y) produces straight lines $x =$ constant as y varies and $y =$ constant as x varies. The plane polar coordinate system (r, θ), on the other hand, produces circles $r =$ constant when θ varies and so is suitable for dealing with circular boundaries in the plane.





Before we attempt to find the solution we must pose the problem *from the beginning* in terms of the coordinates that are appropriate to the boundary conditions. This means, of course, that Laplace's equation must also be given in the same coordinates. To convert Laplace's equation from its current form in Cartesians (x, y) to a new form in plane polar coordinates (r, θ) where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

requires manipulations using Frame 11 onwards of Programme 10. We shall not go into this here, suffice it to say that in plane polar coordinates Laplace's equation is

$$\frac{\partial^2 v(r, \theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r, \theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r, \theta)}{\partial \theta^2} = 0$$

where $v(r, \theta)$ is the expression obtained by changing the coordinates in $u(x, y)$ using $x = r\cos\theta$ and $y = r\sin\theta$.

We shall now pose the problem anew in the next frame

54 The problem

Find the solution to

$$\frac{\partial^2 v(r, \theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r, \theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r, \theta)}{\partial \theta^2} = 0$$

in the circular region $x^2 + y^2 = a^2$ (that is, for $0 \le r \le a$) of the plane where

1. $v(r, \theta)$ is finite for $0 \le r \le a$ and for all θ
2. $v(a, \theta) = f(\theta)$ – the condition on the boundary of the circular region
3. θ is unbounded but $v(r, \theta + 2\pi) = v(r, \theta)$ for $0 \le r \le a$. That is, though θ can take any finite value, the value of $v(r, \theta)$ repeats itself as θ winds round every 2π.

▶

Separating the variables

The variables are r and θ and we assume they are separable and write $v(r,\theta) = R(r)\Theta(\theta)$. This form is then substituted into Laplace's equation and the entire equation multiplied by $\dfrac{r^2}{R(r)\Theta(\theta)}$ to obtain

$$\ldots\ldots\ldots\ldots = 0$$

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$$\boxed{\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} + \frac{1}{\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2} = 0}$$

Because

Substituting $R(r)\Theta(\theta)$ for $v(r,\theta)$ gives

$$\frac{\partial^2 R(r)\Theta(\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)\Theta(\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 R(r)\Theta(\theta)}{\partial\theta^2} = 0$$

That is

$$\Theta(\theta)\frac{d^2R(r)}{dr^2} + \frac{\Theta(\theta)}{r}\frac{dR(r)}{dr} + \frac{R(r)}{r^2}\frac{d^2\Theta(\theta)}{d\theta^2} = 0$$

Multiplying the entire equation by $\dfrac{r^2}{R(r)\Theta(\theta)}$ then gives

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} + \frac{1}{\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2} = 0$$

From this result we can say that

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2} = k$$

which gives rise to the two uncoupled, second-order ordinary differential equations

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} = k \text{ so that}$$

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} = kR(r) \qquad (1)$$

and

$$\frac{1}{\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2} = -k \text{ so that } \frac{d^2\Theta(\theta)}{d\theta^2} = -k\Theta(\theta) \qquad (2)$$

The general solution to equation (2) for $k > 0$ is

$$\ldots\ldots\ldots\ldots$$

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$$\boxed{\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta \quad \text{where } n = 1, 2, \ldots}$$

Because

To solve $\dfrac{d^2\Theta(\theta)}{d\theta^2} = -k\Theta(\theta)$, that is $\dfrac{d^2\Theta(\theta)}{d\theta^2} + k\Theta(\theta) = 0$ we use the auxiliary equation $m^2 + k = 0$ with solutions $m = \pm j\sqrt{k}$. This gives the solution, periodic with period 2π as

$$\Theta(\theta) = A\cos\sqrt{k}\theta + B\sin\sqrt{k}\theta \qquad (3)$$

provided $k > 0$ so that m is pure imaginary. If $k < 0$ then non-periodic solutions would result which would be physically incorrect. To ensure periodicity, that is to ensure that $k > 0$ write $k = n^2$, $n = 1, 2, \ldots$.

$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$ is a solution to equation (2).

We shall look at the case $n = 0$ later.

Substituting $k = n^2$ into equation (1) then gives

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} = n^2R(r) \qquad (4)$$

As a trial solution to equation (4) let $R(r) = pr^q$. Substitution into (4) gives

$q = \ldots\ldots\ldots\ldots$

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$$\boxed{q = \pm n \quad \text{where } n = 1, 2, \ldots}$$

Because

$r\dfrac{dR(r)}{dr} = r\dfrac{d(pr^q)}{dr} = rpqr^{q-1} = pqr^q$. Similarly $r^2\dfrac{d^2R(r)}{dr^2} = pq(q-1)r^q$.

Therefore, substitution into $r^2\dfrac{d^2R(r)}{dr^2} + r\dfrac{dR(r)}{dr} = n^2R(r)$ gives

$[q(q-1) + q]pr^q = n^2pr^q$ and so $[q^2 - n^2]pr^q = 0$ giving $q = \pm n$ where $n = 1, 2, \ldots$

Therefore, a solution to equation (4) is

$R_n(r) = c_nr^n + d_nr^{-n}$ provided $n \neq 0$. The case $n = 0$ is special.

Summary

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To summarise the results so far, we have started to solve Laplace's equation

$$\frac{\partial^2 v(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial v(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v(r,\theta)}{\partial \theta^2}=0$$

in the circular region $x^2+y^2=a^2$ (that is, for $0 \le r \le a$) of the plane where

1 $v(r,\theta)$ is finite for $0 \le r \le a$ and for all θ
2 $v(a,\theta)=f(\theta)$
3 θ is unbounded but $v(r,\theta+2\pi)=v(r,\theta)$ for $0 \le r \le a$.

We have found that, assuming $v(r,\theta)=R(r)\Theta(\theta)$ then, provided $n \neq 0$

$$\Theta_n(\theta)=a_n\cos n\theta+b_n\sin n\theta$$
$$R_n(r)=c_nr^n+d_nr^{-n}$$

So that

$$v_n(r,\theta)=R_n(r)\Theta_n(\theta)=(c_nr^n+d_nr^{-n})(a_n\cos n\theta+b_n\sin n\theta)$$

If we now apply the boundary condition **1** we find that

$$d_n=\ldots\ldots\ldots\ldots$$

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$$\boxed{d_n=0}$$

Because

$v(r,\theta)$ is finite for $0 \le r \le a$. In particular, the solution is finite when $r=0$ and so we cannot have a term of the form r^{-n}. Accordingly $d_n=0$, so omitting the r^{-n} term the solution then becomes

$$v_n(r,\theta)=c_nr^n(a_n\cos n\theta+b_n\sin n\theta)$$

There is an infinite number of such solutions (eigenfunctions), one for each eigenvalue n. The complete solution to Laplace's equation is then a linear combination of all these eigenfunctions. That is

$$v(r,\theta)=\sum_{n=1}^{\infty}e_nv_n(r,\theta)=\sum_{n=1}^{\infty}r^n(A_n\cos n\theta+B_n\sin n\theta)$$

And now for the $n=0$ case

60 The $n=0$ case

When $n=0$ then $k=0$ and equation (1) becomes

$$r^2\frac{d^2R(r)}{dr^2}+r\frac{dR(r)}{dr}=0$$

and if we let $S(r)=\dfrac{dR(r)}{dr}$ then this equation becomes

$$r^2\frac{dS(r)}{dr}+rS(r)=0,\text{ that is } r\left[r\frac{dS(r)}{dr}+S(r)\right]=0 \text{ and so}$$

$$r\frac{dS(r)}{dr}+S(r)=\frac{d[rS(r)]}{dr}=0$$

This has the solution

$$rS(r)=\alpha \text{ (constant) and so } S(r)=\frac{dR(r)}{dr}=\frac{\alpha}{r}$$

$$\text{giving } R(r)=\alpha\ln r+\beta \qquad (5)$$

When $n=0$ then $k=0$ and equation (2) becomes

$$\frac{d^2\Theta(\theta)}{d\theta^2}=0 \text{ with solution } \Theta(\theta)=\gamma\theta+\delta \qquad (6)$$

Applying the boundary conditions to the solutions (5) and (6) gives

$\alpha=$ .... and $\gamma=$ ....

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$$\boxed{\alpha=0 \text{ and } \gamma=0}$$

Because

(a) $v(r,\theta)$ is finite for $0\le r\le a$, in particular when $r=0$, and so $\alpha=0$

(b) $v(r,\theta+2\pi)=v(r,\theta)$. That is, though θ can take any finite value, the value of $v(r,\theta)$ repeats itself as θ winds round every 2π and this means that $\gamma=0$.

So, when $n=0$ the solution is $v_0(r,\theta)=$ constant. We therefore write the complete solution as

$$v(r,\theta)=\frac{A_0}{2}+\sum_{n=1}^{\infty}r^n(A_n\cos n\theta+B_n\sin n\theta)$$

where the constant is taken to be in the form $\dfrac{A_0}{2}$.

Applying the condition on the boundary where $v(a,\theta)=f(\theta)$ we see that

$$f(\theta)=\frac{A_0}{2}+\sum_{n=1}^{\infty}a^n(A_n\cos n\theta+B_n\sin n\theta)$$

which is a Fourier series and hence the form of the constant term being taken as $\dfrac{A_0}{2}$.

▶

The Fourier coefficients are then

$$A_n = \frac{1}{2\pi a^n}\int_0^{2\pi} f(\theta)\cos n\theta \,\mathrm{d}\theta \quad \text{and} \quad B_n = \frac{1}{2\pi a^n}\int_0^{2\pi} f(\theta)\sin n\theta \,\mathrm{d}\theta$$

Example

Solve Laplace's equation

$$\frac{\partial^2 v(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r,\theta)}{\partial \theta^2} = 0$$

in the circular region $x^2 + y^2 = a^2$ of the plane where

1 $v(r, \theta)$ is finite for $0 \le r \le a$ and for all θ
2 $v(a, \theta) = \sin\theta$
3 $v(r, \theta + 2\pi) = v(r, \theta)$ for $0 \le r \le a$.

The solution, as we have seen, is

$$v(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \quad \text{where}$$

$A_n = \ldots\ldots\ldots\ldots$ and $B_n = \ldots\ldots\ldots\ldots$

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$$\boxed{A_n = 0 \quad \text{and} \quad B_n = \frac{1}{2a^n}\delta_{1,n}}$$

Because

$$A_n = \frac{1}{2\pi a^n}\int_0^{2\pi} f(\theta)\cos n\theta \,\mathrm{d}\theta = \frac{1}{2\pi a^n}\int_0^{2\pi} \sin\theta \cos n\theta \,\mathrm{d}\theta = 0 \quad \text{and}$$

$$B_n = \frac{1}{2\pi a^n}\int_0^{2\pi} f(\theta)\sin n\theta \,\mathrm{d}\theta = \frac{1}{2\pi a^n}\int_0^{2\pi} \sin\theta \sin n\theta \,\mathrm{d}\theta = \frac{1}{2\pi a^n}\pi\delta_{1,n}$$

where $\delta_{1,n}$ is the Kronecker delta

That is, $B_1 = \dfrac{1}{2a}$, $B_n = 0$ for $n = 2, 3, \ldots$. The complete solution is then

$$v(r,\theta) = \frac{r}{a}\sin\theta$$

Notice that all three conditions in Frame 61 are satisfied by this solution, that is

1 $v(r,\theta) = \dfrac{r}{a}\sin\theta$ is finite for $0 \le r \le a$ and for all θ

2 $v(a,\theta) = \dfrac{a}{a}\sin\theta = \sin\theta$

3 $v(r, \theta + 2\pi) = \dfrac{r}{a}\sin(\theta + 2\pi) = \dfrac{r}{a}\sin\theta = v(r,\theta)$ for $0 \le r \le a$.

That covers the main steps in the method of solving linear, second-order partial differential equations applied specifically to the wave equation, the heat conduction equation and Laplace's equation. The same approach can be made with other similar equations. ▶

The **Revision summary** and the **Can You?** checklist now follow, then the **Test exercise** with problems like those we have considered. Although the solutions take rather more steps than with other forms of equations, the method is straightforward and follows a clear pattern. The **Further problems** give additional practice.

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Revision summary 11

1 *Ordinary second-order linear differential equations*

(a) Equation of the form $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0$

Auxiliary equation $am^2+bm+c=0$

(1) Real and different roots: $m=m_1$ and $m=m_2$

$$y=Ae^{m_1x}+Be^{m_2x}$$

(2) Real and equal roots: $m=m_1$ (twice)

$$y=e^{m_1x}(A+Bx)$$

(3) Complex roots: $m=\alpha\pm j\beta$

$$y=e^{\alpha x}\{A\cos\beta x+B\sin\beta x\}.$$

(b) Equations of the form $\dfrac{d^2y}{dx^2}\pm n^2y=0$

(1) $\dfrac{d^2y}{dx^2}+n^2y=0;\quad y=A\cos nx+B\sin nx$

(2) $\dfrac{d^2y}{dx^2}-n^2y=0;\quad y=A\cosh nx+B\sinh nx$

or $\quad y=Ae^{nx}+Be^{-nx}$

or $\quad y=A\sinh n(x+\phi)$.

2 *Partial differential equations* Solution $u=f(x,y,t,\ldots)$

Linear equations: If $u=u_1$, $u=u_2$, $u=u_3$, … are solutions, so also is $u=u_1+u_2+u_3+\ldots+u_r+\ldots=\sum_{r=1}^{\infty}u_r$.

(a) *Wave equation* – transverse vibrations of an elastic string

$$\frac{\partial^2u}{\partial x^2}=\frac{1}{c^2}\cdot\frac{\partial^2u}{\partial t^2}\quad\text{where } c^2=\frac{T}{\rho},\quad T=\text{tension of string}$$

$\rho=$ mass per unit length.

(b) *Heat conduction equation* – heat flow in uniform finite bar

$$\frac{\partial^2u}{\partial x^2}=\frac{1}{c^2}\cdot\frac{\partial u}{\partial t}\qquad\text{where } c^2=\frac{k}{\sigma\rho}$$

$k=$ thermal conductivity of material

$\sigma=$ specific heat of the material

$\rho=$ mass per unit length of bar.

▶

(c) *Laplace equation* – distribution of a field over a plane area

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

3 *Separating the variables*
Let $u(x, y)=X(x)Y(y)$ where $X(x)$ is a function of x only
and $Y(y)$ is a function of y only.

Then $\frac{\partial u}{\partial x}=X'Y$; $\frac{\partial^2 u}{\partial x^2}=X''Y$

$\frac{\partial u}{\partial y}=XY'$; $\frac{\partial^2 u}{\partial y^2}=XY''$

Substitute in the given partial differential equation and form separate differential equations to give $X(x)$ and $Y(y)$ by introducing a common constant $(-p^2)$. Determine arbitrary functions by use of the initial and boundary conditions.

4 *Laplace's equation in plane polar coordinates*

$$\frac{\partial^2 v(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial v(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v(r,\theta)}{\partial \theta^2}=0$$

Separating the variables by $v(r,\theta)=R(r)\Theta(\theta)$ produces two uncoupled, second-order ordinary differential equations

$$r^2\frac{d^2R(r)}{dr^2}+r\frac{dR(r)}{dr}=kR(r)$$

and $$\frac{d^2\Theta(\theta)}{d\theta^2}=-k\Theta(\theta)$$

These two ordinary differential equations can then be solved under the application of appropriate boundary conditions.

Can You?

Checklist 11

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Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: **Frames**

- Summarise the introductory methods of solving ordinary differential equations? 1

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve partial differential equations that are amenable to solution by direct integration? 2 to 7

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Apply initial and boundary conditions? 5 to 7

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve the one-dimensional wave and heat equations by separating the variables and obtaining eigenfunctions and corresponding eigenvalues? 8 to 40

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve the two-dimensional Laplace equation in Cartesian coordinates? 41 to 52

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise the need for alternative coordinate systems and solve the two-dimensional Laplace equation in plane polar coordinates? 53 to 62

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 11

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1 Solve the following equations

(a) $\dfrac{\partial^2 u}{\partial x^2} = 24x^2(t-2)$, given that at $x = 0$, $u = e^{2t}$ and $\dfrac{\partial u}{\partial x} = 4t$.

(b) $\dfrac{\partial^2 u}{\partial x \partial y} = 4e^y \cos 2x$, given that at $y = 0$, $\dfrac{\partial u}{\partial x} = \cos x$ and at $x = \pi$, $u = y^2$.

2 A perfectly elastic string is stretched between two points 10 cm apart. Its centre point is displaced 2 cm from its position of rest at right angles to the original direction of the string and then released with zero velocity. Applying the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$ with $c^2 = 1$, determine the subsequent motion $u(x, t)$.

3 One end A of an insulated metal bar AB of length 2 m is kept at 0°C while the other end B is maintained at 50°C until a steady state of temperature along the bar is achieved. At $t = 0$, the end B is suddenly reduced to 0°C and kept at that temperature. Using the heat conduction equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial u}{\partial t}$, determine an expression for the temperature at any point in the bar distance x from A at any time t.

▶

4 A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 2$, $y = 2$. Apply the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ to determine the potential distribution $u(x, y)$ over the plate, subject to the following boundary conditions.

$$\begin{array}{lll} u = 0 & \text{when } x = 0 & 0 \le y \le 2 \\ u = 0 & \text{when } x = 2 & 0 \le y \le 2 \\ u = 0 & \text{when } y = 0 & 0 \le x \le 2 \\ u = 5 & \text{when } y = 2 & 0 \le x \le 2. \end{array}$$

5 Solve Laplace's equation in plane polar coordinates

$$\frac{\partial^2 v(r, \theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r, \theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r, \theta)}{\partial \theta^2} = 0$$

in the circular region $x^2 + y^2 = 1$ of the plane where

(1) $v(r, \theta)$ is finite for $0 \le r \le 1$ and for all θ
(2) $v(1, \theta) = 5\cos 3\theta$
(3) $v(r, \theta + 2\pi) = v(r, \theta)$ for $0 \le r \le 1$.

Further problems 11

66

1 Show that the equation $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2} = 0$ is satisfied by $u = f(x + ct) + F(x - ct)$ where f and F are arbitrary functions.

2 If $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$ and $c = 3$, determine the solution $u = f(x, t)$ subject to the boundary conditions

$$u(0, t) = 0 \text{ and } u(2, t) = 0 \text{ for } t \ge 0$$

$$u(x, 0) = x(2 - x) \text{ and } \left[\frac{\partial u}{\partial t}\right]_{t=0} = 0 \qquad 0 \le x \le 2.$$

3 The centre point of a perfectly elastic string stretched between two points A and B, 4 m apart, is deflected a distance 0·01 m from its position of rest perpendicular to AB and released initially with zero velocity. Apply the wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$ where $c = 10$ to determine the subsequent motion of a point P distant x from A at time t.

4 An elastic string is stretched between two points 10 cm apart. A point P on the string 2 cm from the left-hand end, i.e. the origin, is drawn aside 1 cm from its position of rest and released with zero velocity. Solve the one-dimensional wave equation to determine the displacement of any point at any instant.

5 An insulated uniform metal bar, 10 units long, has the temperature of its ends maintained at 0°C and at $t = 0$ the temperature distribution $f(x)$ along the bar is defined by $f(x) = x(10 - x)$. Solve the heat conduction equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial u}{\partial t}$ with $c^2 = 4$ to determine the temperature u of any point in the bar at time t.

6 The ends of an insulated rod AB, 10 units long, are maintained at 0°C. At $t = 0$, the temperature within the rod rises uniformly from each end reaching 2°C at the mid-point of AB. Determine an expression for the temperature $u(x, t)$ at any point in the rod, distant x from the left-hand end at any subsequent time t.

7 A rectangular plate OPQR is bounded by the lines $x = 0$, $y = 0$, $x = 4$, $y = 2$. Determine the potential distribution $u(x, y)$ over the rectangle using the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, subject to the following boundary conditions

$$\begin{array}{ll} u(0, y) = 0 & 0 \leq y \leq 2 \\ u(4, y) = 0 & 0 \leq y \leq 2 \\ u(x, 2) = 0 & 0 \leq x \leq 4 \\ u(x, 0) = x(4 - x) & 0 \leq x \leq 4. \end{array}$$

8 Two sides AB and AD of a rectangular plate ABCD lie along the x and y axes respectively. The remaining two sides are the lines $x = 5$ and $y = 2$. The sides BC, CD and DA are maintained at zero temperature. The temperature distribution along AB is defined by $f(x) = x(x - 5)$. Determine an expression for the steady-state temperature at any point in the plate.

9 Solve Laplace's equation in plane polar coordinates

$$\frac{\partial^2 v(r, \theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r, \theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r, \theta)}{\partial \theta^2} = 0$$

in the circular region $x^2 + y^2 = 1$ of the plane where

(1) $v(r, \theta)$ is finite for $0 \leq r \leq 1$ and for all θ
(2) $v(1, \theta) = \sin 2\theta - 4\cos\theta$
(3) $v(r, \theta + 2\pi) = v(r, \theta)$ for $0 \leq r \leq 1$.

10 Solve Laplace's equation in plane polar coordinates

$$\frac{\partial^2 v(r, \theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r, \theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r, \theta)}{\partial \theta^2} = 0$$

in the circular region $x^2 + y^2 = 1$ of the plane where

(1) $v(r, \theta)$ is finite for $0 \leq r \leq 1$ and for all θ
(2) $v(1, \theta) = 3\sin^2\theta$
(3) $v(r, \theta + 2\pi) = v(r, \theta)$ for $0 \leq r \leq 1$.